Solid state systems for quantum information, Correction 9

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Exercise 1 : Jaynes-Cumming model: spectrum, entanglement and dynamics

In this exercise we are going to study the fundamental properties of the Jaynes-Cumming model, whose Hamiltonian in the rotating wave approximation is written as follows ($\hbar = 1$)

$$\hat{H} = \frac{\omega_q}{2}\hat{\sigma}^z + \omega_r \hat{a}^\dagger \hat{a} + g(\hat{a}^\dagger \hat{\sigma}^- + \hat{a}\hat{\sigma}^+). \quad (1)$$

The Jaynes-Cumming model describes the interaction between a quantized mode of the electromagnetic field and a two-level atom, and it is one of the paradigmatic models in quantum optics. In Eq. (39), ω_q is the qubit frequency, ω_r is the resonator frequency, and g is the light-matter coupling. The frequency distance between the two-level system and the resonator is called detuning, $\Delta = \omega_q - \omega_r$. We are going to study this model systematically, from its fundamental symmetries to the exact spectrum and the dynamics.

1. Determine the symmetries of the Hamiltonian. The eigenvectors of the non-interacting Hamiltonian ($g = 0$) are tensor products between the $\hat{\sigma}^z$ ground and excited states, and the Fock states, $\mathcal{H} = \text{span}\{|g, e\rangle \otimes |n\rangle_{n\in\mathbb{N}}\}$. These states are also called the bare eigenvectors. We will consider this basis throughout the exercise. Find the symmetry of $\hat{H}$ and the Hermitian operator $\hat{O}$ such that $[\hat{H}, \hat{O}] = 0$ (*i.e.*, $\hat{O}$ is a conserved quantity of the system). Write $\hat{O}$ in the eigenbasis of the non-interacting problem and in terms of the bosonic creation and annihilation operators and of $\hat{\sigma}^z$.

2. Find the exact spectrum of the Jaynes-Cumming model. To do this:

   (a) Compute the matrix elements of the interacting term (proportional to g) and show that $\hat{H}$ is block-diagonal with 2×2 blocks. Interpret this finding in light of what you found in the previous point.

   (b) Diagonalize the 2×2 block finding eigenvalues and eigenvectors of $\hat{H}$. Write the eigenvectors in terms of the bare eigenvectors. These eigenvectors are called dressed eigenvectors. Hint: To find the eigenvectors, use the rotation matrix (also called Bogoliubov matrix)

   $$U = \begin{pmatrix} \sin(\theta_n/2) & \cos(\theta_n/2) \\ \cos(\theta_n/2) & -\sin(\theta_n/2) \end{pmatrix} \quad (2)$$

   And compute the angle θ_n.

3. Show that the dressed states are generically entangled. Moreover, show that $|\Delta| = 0$ corresponds to maximal entanglement whereas $|\Delta| \to \infty$ gives back a product state. Provide an intuitive explanation about this finding.
   Hint: To prove that a quantum state $|\Psi\rangle$ of a bipartite system $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ is entangled in

some basis which is the tensor product of two single-particle bases, $|\phi\rangle = |a\rangle \otimes |b\rangle$, you need to show that the reduced density matrix describing one of the two subsystem is not pure. To compute the reduced density matrix, you need to perform the partial trace, which amounts to tracing out one of the two subsystems:

$$\hat{\rho}_a = \mathrm{Tr}_b(\hat{\rho}) = \sum_b \langle b|\,\Psi\rangle\langle\Psi\,|b\rangle\,. \tag{3}$$

To asses whether $\hat{\rho}_a$ is pure or not, you can compute the purity $\gamma_a = \mathrm{Tr}(\hat{\rho}_a^2)$. If $\gamma_a < 1$ then the state $|\Psi\rangle$ is entangled. If $\gamma = 1/\dim(\mathcal{H}_a)$, then the system is maximally entangled.

4. We now analyze the Jaynes-Cumming dynamics. To solve for the dynamics, proceed with the following steps:
   (a) Write a single 2×2 block of the Hamiltonian $\hat{H}$ in terms of the Pauli matrices
   (b) Rewrite the time evolution operator in the form $\exp\left[i\frac{\Omega_n t}{2}(n_x\hat{\sigma}^x + n_z\hat{\sigma}^z)\right]$, with $n_x^2 + n_z^2 = 1$. Calculate Ω_n, n_z and n_x. How is Ω_n called?
   (c) Rewrite the above results in terms of a 2×2 matrix.
   (d) Starting with the state $|\psi(0)\rangle = |g, n+1\rangle$ calculate the state $|\psi(t)\rangle$ at time t. Why is it sufficient to consider the time evolution of small blocks, instead of the time evolution of the full $\hat{H}$?
5. Finally, we study atomic inversion in the Jaynes-Cumming model. Use your results from the previous question to calculate the atomic inversion, given as $w(t) = |C_e(t)|^2 - |C_g(t)|^2$ when starting in the state $|\Psi(0)\rangle = |g, n+1\rangle$ with $C_g(t) = \langle g, n+1|\Psi(t)\rangle$ and $C_e(t) = \langle e, n|\Psi(t)\rangle$. What is the atomic inversion, if the detuning is $\Delta = 0$?

Solution 1 :

1. The Hamiltonian
$$\hat{H} = \frac{\omega_q}{2}\hat{\sigma}^z + \omega_r\hat{a}^\dagger\hat{a} + g(\hat{a}^\dagger\hat{\sigma}^- + \hat{a}\hat{\sigma}^+), \tag{4}$$
conserves the total number of excitations. Indeed, the unperturbed part of the Hamiltonian, $\omega_q\hat{\sigma}^z/2 + \omega_r\hat{a}^\dagger\hat{a}$, does not change the state of the qubit ($|g\rangle$ or $|e\rangle$) or the photon number in the cavity (a Fock state $|n\rangle$). The perturbation $g(\hat{a}^\dagger\hat{\sigma}^- + \hat{a}\hat{\sigma}^+)$, instead, conserves the *total* number of excitations: if an excitation is lost by the resonator, the qubit gains the excitation and viceversa. The symmetry associated with the conservation of the total number of particles is the $\mathbb{U}(1)$ symmetry and the symmetry operator is the particle-number operator, $\hat{N}$, which we can write as
$$\hat{N} = \sum_n n\,|n\rangle\langle n| + |e\rangle\langle e| = \hat{a}^\dagger\hat{a} + \frac{1}{2}(-\hat{\sigma}^z + \mathbb{1}). \tag{5}$$
The first term counts the photon number in the resonator, the second term monitors whether the qubit is excited. Let's prove that $\hat{N}$ is a conserved quantity:
$$[\hat{N}, \hat{H}] = [\hat{N}, g(\hat{a}^\dagger\hat{\sigma}^- + \hat{a}\hat{\sigma}^+)] = g[\hat{N}, \hat{a}\hat{\sigma}^+] + g[\hat{N}, \hat{a}^\dagger\hat{\sigma}^-], \tag{6}$$

where we used the fact that $\omega_q\hat{\sigma}^z/2 + \omega_r\hat{a}^\dagger\hat{a}$ trivially commutes with $\hat{N}$. Now we have

$$[\hat{a}^\dagger\hat{a} - \hat{\sigma}^z/2, \hat{a}\hat{\sigma}^+] = -\hat{a}\hat{\sigma}^+ + \hat{a}\hat{\sigma}^+ = 0, \tag{7}$$

and

$$[\hat{a}^\dagger\hat{a} - \hat{\sigma}^z/2, \hat{a}\hat{\sigma}^+] = \hat{a}^\dagger\hat{\sigma}^- - \hat{a}^\dagger\hat{\sigma}^- = 0. \tag{8}$$

From this we deduce that $[\hat{N}, \hat{H}] = 0$, and $\hat{N}$ is a conserved quantity in the system.

2. We established that the Jaynes-Cumming Hamiltonian conserves the total number of excitations, *i.e.*, the system is characterized by a $\mathbb{U}(1)$ symmetry. A quantum symmetry allows the block diagonalization of the Hamiltonian, each one labeled by a quantum number, given by the symmetry operator. The symmetry blocks can be constructed by computing the matrix elements of $\hat{H}$ over the states which respect the symmetry. If the symmetry is $\mathbb{U}(1)$, then to construct the symmetry block with $n+1$ excitations, one has to collect the states describing $n+1$ excitations.

   (a) For the Jaynes-Cumming model, there are only two states which describe $n+1$ excitations: qubit down and $n+1$ photons in the resonator, $|g, n+1\rangle$; qubit up and n photons in the resonator, $|e, n\rangle$. We have then

$$\begin{aligned}
\langle g, n+1|\,\hat{H}\,|g, n+1\rangle &= -\frac{\omega_q}{2} + \omega_r(n+1), \\
\langle e, n|\,\hat{H}\,|e, n\rangle &= \frac{\omega_q}{2} + \omega_r n, \\
\langle g, n+1|\,\hat{H}\,|e, n\rangle &= g\,\langle g, n+1|\,\hat{a}^\dagger\hat{\sigma}^-\,|e, n\rangle = g\sqrt{n+1}, \\
\langle e, n|\,\hat{H}\,|g, n+1\rangle &= g\,\langle e, n|\,\hat{a}\hat{\sigma}^+\,|g, n+1\rangle = g\sqrt{n+1}.
\end{aligned} \tag{9}$$

   The symmetry block thus reads

$$M = \begin{pmatrix} -\frac{\omega_q}{2} + \omega_r(n+1) & g\sqrt{n+1} \\ g\sqrt{n+1} & \frac{\omega_q}{2} + \omega_r n \end{pmatrix}. \tag{10}$$

   (b) To diagonalize the symmetry block, we first introduce the detuning $\Delta = \omega_q - \omega_r$ and we rewrite the symmetry block as

$$M = \omega_r\left(n + \frac{1}{2}\right)\mathbb{1} + \frac{1}{2}\begin{pmatrix} -\Delta & 2g\sqrt{n+1} \\ 2g\sqrt{n+1} & \Delta \end{pmatrix} = \omega_r\left(n + \frac{1}{2}\right)\mathbb{1} + M'. \tag{11}$$

   To find the eigenvalues, it is sufficient to solve the secular equation

$$\det\left(M' - \varepsilon I\right) = \begin{vmatrix} -\Delta - \varepsilon & 2g\sqrt{n+1} \\ 2g\sqrt{n+1} & \Delta - \varepsilon \end{vmatrix} = 0, \tag{12}$$

   which yields the equation

$$\varepsilon^2 - \Delta^2 - 4g^2(n+1) = 0, \tag{13}$$

   whose solutions read

$$\varepsilon_\pm = \pm\sqrt{4g^2(n+1) + \Delta^2}. \tag{14}$$

The dressed eigenenergies finally reads

$$E_{n,\pm} = \omega_r \left(n + \frac{1}{2}\right) \pm \frac{1}{2}\sqrt{4g^2(n+1) + \Delta^2}. \tag{15}$$

Now we find the eigenvectors. The term proportional to $\omega_r(n+1/2)$ gives only a global shift to the Hamiltonian and does not affect the eigenvectors, which are given by the eigenvectors of M'. To find the eignevcetors, we use the Bogoliubov matrix U given in the text and we need to compute $U^\dagger M' U$

$$\begin{pmatrix} \sin(\theta_n/2) & \cos(\theta_n/2) \\ \cos(\theta_n/2) & -\sin(\theta_n/2) \end{pmatrix} \begin{pmatrix} -a & b \\ b & a \end{pmatrix} \begin{pmatrix} \sin(\theta_n/2) & \cos(\theta_n/2) \\ \cos(\theta_n/2) & -\sin(\theta_n/2) \end{pmatrix}, \tag{16}$$

where $a = \Delta/2$ and $b = g\sqrt{n+1}$. The matrix is diagonal when the off-diagonal elements in the above matrix product are zero. By carrying out the matrix multiplication, one arrives at the condition

$$\begin{aligned} 0 &= -a\sin(\theta_n/2)\cos(\theta_n/2) + b\cos^2(\theta_n/2) - b\sin^2(\theta_n/2) - a\cos(\theta_n/2)\sin(\theta_n/2) \\ &= -a\sin(\theta_n) + b\cos(\theta_n), \end{aligned} \tag{17}$$

which leads to $\tan(\theta_n) = b/a = 2g\sqrt{n+1}/\Delta$. The eigenvectors are therefore

$$\begin{aligned} |n,+\rangle &= \sin\left(\frac{\theta_n}{2}\right)|g,n+1\rangle + \cos\left(\frac{\theta_n}{2}\right)|e,n\rangle, \\ |n,-\rangle &= \cos\left(\frac{\theta_n}{2}\right)|g,n+1\rangle - \sin\left(\frac{\theta_n}{2}\right)|e,n\rangle. \end{aligned} \tag{18}$$

3. We focus our analysis on $|n,+\rangle$. The associated density matrix is

$$\hat{\rho} = |n,+\rangle\langle n,+| = \sin^2\left(\frac{\theta_n}{2}\right)|g,n+1\rangle\langle g,n+1| \tag{19}$$

$$+ \sin\left(\frac{\theta_n}{2}\right)\cos\left(\frac{\theta_n}{2}\right)[|g,n+1\rangle\langle e,n| + |e,n\rangle\langle g,n+1|] + \cos^2\left(\frac{\theta_n}{2}\right)|e,n\rangle\langle e,n|. \tag{20}$$

The qubit's reduced density matrix is obtained by performing the trace over the bosonic Fock space

$$\hat{\rho}_q = \mathrm{Tr}_n(\hat{\rho}) = \sum_n \langle n|\hat{\rho}|n\rangle = \sin^2\left(\frac{\theta_n}{2}\right)|g\rangle\langle g| + \cos^2\left(\frac{\theta_n}{2}\right)|e\rangle\langle e|. \tag{21}$$

Finally, the purity is

$$\gamma = \mathrm{Tr}(\hat{\rho}_q^2) = \sin^4\left(\frac{\theta_n}{2}\right) + \cos^4\left(\frac{\theta_n}{2}\right) \leq 1. \tag{22}$$

We conclude that the dressed state $|n,+\rangle$ is an entangled state.

If $\Delta = 0$, then $\theta_n = \mathrm{atan}\big(2g\sqrt{n+1}/\Delta\big) = \pi/2$ and $\gamma = 1/4 + 1/4 = 1/2$. Therefore, at $\Delta = 0$ the qubit is maximally entangled with the resonator. If $\Delta \to \infty$, then $\theta_n = 0$ and $\gamma = 1 + 0 = 1$. Therefore at $\Delta \to \infty$ the qubit is in a pure state and it is not entangled with the resonator.

4. We now study the dynamics of the Jaynes-Cumming model.

(a) The Jaynes-Cumming block can be rewritten in terms of the Pauli matrices as

$$M = \omega_r \left(n + \frac{1}{2}\right) \mathbb{1} - \frac{\Delta}{2}\hat{\sigma}^z + g\sqrt{n+1}\hat{\sigma}^x = \omega_r \left(n + \frac{1}{2}\right) \mathbb{1} + \hat{H}_{\mathrm{TLS}}. \tag{23}$$

Notice that the term proportional to $\omega_r(n + 1/2)$ is a constant and does not participate to the dynamics. All the dynamics is generated by $\hat{H}_{\mathrm{TLS}}$.

(b) Let's us first write down the time evolution of $\hat{H}_{\mathrm{TLS}}$

$$e^{-i\hat{H}_{\mathrm{TLS}}t} = e^{-2it\left(-\Delta\hat{\sigma}^z + 2g\sqrt{n+1}\hat{\sigma}^x\right)}. \tag{24}$$

By comparing it with the given form $e^{i\frac{\Omega_n t}{2}(n_z\hat{\sigma}^z + n_x\hat{\sigma}^x)}$, we have

$$\Omega_n(n_x\hat{\sigma}^x + n_z\hat{\sigma}^z) = -2g\sqrt{n+1}\hat{\sigma}^x + \Delta\hat{\sigma}^z \tag{25}$$

and we obtain

$$n_x = -\frac{2g\sqrt{n+1}}{\Omega_n} \qquad n_z = \frac{\Delta}{\Omega_n} \tag{26}$$

Now, using the condition $n_x^2 + n_z^2 = 1$ one has

$$\frac{4g^2(n+1)}{\Omega_n^2} + \frac{\Delta^2}{\Omega_n^2} = 1 \quad \Longrightarrow \quad \Omega_n = \sqrt{\Delta^2 + 4g^2(n+1)}, \tag{27}$$

Here Ω_n is called Rabi frequency.

(c) Since $n_x^2 + n_z^2 = 1$, the following identity can be used to express the results:

$$e^{i\frac{\theta}{2}(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z)} = \cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)(n_x\hat{\sigma}_x + n_y\hat{\sigma}_y + n_z\hat{\sigma}_z). \tag{28}$$

This leads to

$$\begin{aligned} e^{-i\hat{H}_{\mathrm{TLS}}t} &= \cos\left(\frac{\Omega_n t}{2}\right) + i\sin\left(\frac{\Omega_n t}{2}\right)(n_x\hat{\sigma}_x + n_z\hat{\sigma}_z) \\ &= \begin{pmatrix} \cos\left(\frac{\Omega_n t}{2}\right) + in_z\sin\left(\frac{\Omega_n t}{2}\right) & in_x\sin\left(\frac{\Omega_n t}{2}\right) \\ in_x\sin\left(\frac{\Omega_n t}{2}\right) & \cos\left(\frac{\Omega_n t}{2}\right) - in_z\sin\left(\frac{\Omega_n t}{2}\right) \end{pmatrix}. \end{aligned} \tag{29}$$

(d) We start with the state $|\Psi(0)\rangle = |g, n+1\rangle$. Let us denote $|0\rangle = |g, n+1\rangle$ and $|1\rangle = |e, n\rangle$. Then $|\Psi(t)\rangle$ is given by

$$\begin{aligned} |\Psi(t)\rangle = e^{-i\hat{H}_{\mathrm{TLS}}t}|\Psi(0)\rangle &= \begin{pmatrix} \cos\left(\frac{\Omega_n t}{2}\right) + in_z\sin\left(\frac{\Omega_n t}{2}\right) & in_x\sin\left(\frac{\Omega_n t}{2}\right) \\ in_x\sin\left(\frac{\Omega_n t}{2}\right) & \cos\left(\frac{\Omega_n t}{2}\right) - in_z\sin\left(\frac{\Omega_n t}{2}\right) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \left[\cos\left(\frac{\Omega_n t}{2}\right) + in_z\sin\left(\frac{\Omega_n t}{2}\right)\right]|g, n+1\rangle + \left[in_x\sin\left(\frac{\Omega_n t}{2}\right)\right]|e, n\rangle. \end{aligned} \tag{30}$$

5. From the previous point we can easily compute the overlaps between the wave function and the bare eigenstates. We have

$$C_g(t) = \langle g, n+1|\Psi(t)\rangle = \cos\left(\frac{\Omega_n t}{2}\right) + i\frac{\Delta}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right) \tag{31}$$
$$C_e(t) = \langle e, n|\Psi(t)\rangle = -i\frac{2g\sqrt{n+1}}{\Omega_n}\sin\left(\frac{\Omega_n t}{2}\right).$$

The atomic inversion is defined as $w(t) = |C_e(t)|^2 - |C_g(t)|^2$ and can then be calculated as follows:

$$\begin{aligned} w(t) &= \frac{4g^2(n+1)}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right) - \cos^2\left(\frac{\Omega_n t}{2}\right) + \frac{\Delta^2}{\Omega_n^2}\sin^2\left(\frac{\Omega_n t}{2}\right) \\ &= \left[\frac{\Delta^2 + 4g^2(n+1)}{\Omega_n^2}\right]\sin^2\left(\frac{\Omega_n t}{2}\right) - \cos^2\left(\frac{\Omega_n t}{2}\right). \end{aligned} \tag{32}$$

For zero detuning ($\Delta = 0$) we have $\Omega_n = \sqrt{4g^2(n+1)}$ and the atomic inversion,

$$w(t) = \sin^2\frac{\Omega_n t}{2} - \cos^2\left(\frac{\Omega_n t}{2}\right). \tag{33}$$

Exercise 2 : Schrieffer-Wolff transformation for transmon readout

In this exercise we are going to apply degenerate perturbation theory to describe the transmon qubit readout. The method we adopt is the Schrieffer-Wolff transformation. Consider the Hamiltonian

$$\hat{H} = \hat{H}_0 + \hat{V}, \tag{34}$$

where $\hat{H}_0$ is some unperturbed Hamiltonian of which we know the spectrum, $\hat{H}_0 \left|\Psi_n\right\rangle = E_n \left|\Psi_n\right\rangle$, $\hat{V}$ is an off-diagonal perturbation. The idea of the Schrieffer-Wolff transformation comes from the Baker-Campbell-Haussdorff expansion: we perform a unitary transformation on $\hat{H}$ with generator $\hat{S}$. Up to second order, we have

$$e^{\hat{S}}\hat{H}e^{-\hat{S}} = \hat{H} + [\hat{S}, \hat{H}] + \frac{1}{2}[\hat{S}, [\hat{S}, \hat{H}]]. \tag{35}$$

Now if we write $\hat{H} = \hat{H}_0 + \hat{V}$ we obtain

$$e^{\hat{S}}\hat{H}e^{-\hat{S}} = \hat{H}_0 + \hat{V} + [\hat{S}, \hat{H}_0] + [\hat{S}, \hat{V}] + \frac{1}{2}[\hat{S}, [\hat{S}, \hat{H}_0]] + \frac{1}{2}[\hat{S}, [\hat{S}, \hat{V}]]. \tag{36}$$

We now impose that the generator $\hat{S}$ is such that $\hat{V} = -[\hat{S}, \hat{H}_0]$, *i.e.*, it cancels the contribution of $\hat{V}$ at the first order. This leads to the second-order Schrieffer-Wolff formula

$$\hat{H}_{\text{eff}} = e^{\hat{S}}\hat{H}e^{-\hat{S}} \simeq \hat{H}_0 + \frac{1}{2}[\hat{S}, \hat{V}]. \tag{37}$$

The problem is of course finding $\hat{S}$. We state here, without proving it, that the Schrieffer-Wolff generator at first order is given by

$$\hat{S} = \sum_{n,m} \frac{\left\langle\Psi_n\right| \hat{V} \left|\Psi_m\right\rangle}{E_n - E_m} \left|\Psi_n\right\rangle \left\langle\Psi_m\right|. \tag{38}$$

The above two equations provide all the ingredients to compute low-energy effective Hamiltonians. We apply this formalism to two examples, both relevant for circuit QED.

1. Jaynes-Cumming model. Consider the Hamiltonian

$$\hat{H}_{\text{JC}} = \frac{\omega_q}{2}\hat{\sigma}^z + \omega_r \hat{a}^\dagger \hat{a} + g(\hat{a}^\dagger \hat{\sigma}^- + \hat{a}\hat{\sigma}^+). \tag{39}$$

   Suppose $g \ll |\Delta| = |\omega_q - \omega_r|$. Compute the effective low-energy Hamiltonian by means of a second-order Schrieffer-Wolff transformation. Justify the use of perturbation theory in this context and give a physical interpretation about the terms appearing in the $\hat{H}_{\text{eff, JC}}$ you find.

2. Dispersive readout of a superconducting transmon qubit. In the previous point, we modeled the transmon qubit as a two-level system. We now want to go beyond this (very) simple approximation and we want to take into account the multilevel structure of the transmon. Consider the Hamiltonian

$$\hat{H}_{\text{cQED}} = \omega_r \hat{a}^\dagger \hat{a} + \omega_q \hat{b}^\dagger \hat{b} - \frac{E_c}{2}\hat{b}^{\dagger 2}\hat{b}^2 + g(\hat{a}^\dagger \hat{b} + \hat{b}^\dagger \hat{a}), \tag{40}$$

where $\hat{a}$ and $\hat{b}$ are the bosonic modes of the resonator and of the transmon, respectively. Suppose $g \ll |\Delta| = |\omega_q - \omega_r|$. Compute the effective low-energy Hamiltonian $H_{\text{eff, cQED}}$ by means of a second-order Schrieffer-Wolff transformation, and neglect possible counter-rotating terms.

3. Starting from $H_{\text{eff, cQED}}$, truncate all the energy levels but the first two and compute the dispersive shift χ (half of the resonator's energy different when the qubit is up or down, respectively). Compare it to the Hamiltonian $\hat{H}_{\text{eff, JC}}$ you found in the previous point. Why are they not coinciding?

Solution 2 :

1. The Jaynes-Cumming Hamiltonian reads:

$$\hat{H} = \omega_r \hat{a}^\dagger \hat{a} + \frac{\omega_q}{2}\hat{\sigma}_z - g(\hat{a}\hat{\sigma}_+ + \hat{a}^\dagger \hat{\sigma}_-). \tag{41}$$

The first two terms are the unperturbed (free) Hamiltonian $\hat{H}_0$ with eigenstates $|n, \sigma\rangle = |n\rangle \otimes |\sigma\rangle$ with $n \in \mathbb{N}$ and $\sigma \in \{1, -1\}$. The last term is the interaction term which describes the coherent exchange of excitations between the harmonic oscillator states and the two-level system states. The effective Hamiltonian in the regime $g \ll |\Delta|$ can be derived by using the Schrieffer-Wolff perturbation theory. Calling $\hat{V} = \hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-$ we derive the effective second-order SW Hamiltonian. First we compute the generator using the formula given in the text. We have

$$\hat{S} = \sum_{m,\sigma'} \sum_{n,\sigma} \frac{\langle m, \sigma'| \hat{V} |n, \sigma\rangle}{E_{m,\sigma'} - E_{n,\sigma}} |m, \sigma'\rangle \langle n, \sigma| . \tag{42}$$

We must compute the matrix elements of $\hat{V}$ in the unperturbed eigenbasis. We get

$$\begin{aligned} \langle m, \sigma'| \hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_- |n, \sigma\rangle &= \langle m, \sigma'| \left(\sqrt{n}\delta_{\sigma,-1} |n-1, -\sigma\rangle + \sqrt{n+1}\delta_{\sigma,+1} |n+1, -\sigma\rangle\right) \\ &= \sqrt{n}\delta_{\sigma,-1}\delta_{m,n-1}\delta_{\sigma',-\sigma} + \sqrt{n+1}\delta_{\sigma,+1}\delta_{m,n+1}\delta_{\sigma',-\sigma}, \end{aligned} \tag{43}$$

so that

$$\hat{S} = \sum_n \frac{\sqrt{n}\,|n-1\rangle\langle n|}{E_{n-1,1} - E_{n,-1}} |1\rangle\langle -1| + \sum_n \frac{\sqrt{n+1}\,|n+1\rangle\langle n|}{E_{n+1,-1} - E_{n,1}} |-1\rangle\langle 1| = \frac{1}{\Delta}(\hat{a}\hat{\sigma}_+ - \hat{a}^\dagger\hat{\sigma}_-). \tag{44}$$

Now have to compute the commutator $[\hat{S}, \hat{V}]$

$$\hat{H}_{\text{eff, JC}} = \omega_r \hat{a}^\dagger \hat{a} + \frac{\omega_q}{2}\hat{\sigma}_z + \frac{g^2}{2\Delta}[\hat{a}\hat{\sigma}_+ - \hat{a}^\dagger\hat{\sigma}_-, \hat{a}\hat{\sigma}_+ + \hat{a}^\dagger\hat{\sigma}_-]. \tag{45}$$

Given the relations $[\hat{a}, \hat{a}^\dagger] = \mathbb{1}$, $[\hat{\sigma}_+, \hat{\sigma}_-] = \hat{\sigma}_z$, $\{\hat{\sigma}_+, \hat{\sigma}_-\} = \mathbb{1}$, we easily obtain the dispersive Jaynes-Cumming Hamiltonian

$$\hat{H}_{\text{eff, JC}} = \omega_r \hat{a}^\dagger \hat{a} + \frac{1}{2}\left(\omega_q + \frac{g^2}{\Delta}\right)\hat{\sigma}_z + \frac{g^2}{\Delta}\hat{a}^\dagger\hat{a}\hat{\sigma}_z. \tag{46}$$

In this limit cavity and qubit are dispersively coupled an measuring the photon number does not change the state of the qubit. Thus, if we measure the resonance frequency of the cavity, we know the state of the qubit. In the above equation, the term $(g^2/\Delta)\hat{a}^\dagger\hat{a}\hat{\sigma}_z$ is the AC Stark shift and increases with the photon number in the cavity. The shift $(g^2/\Delta)\hat{\sigma}_z$ is instead the Lamb shift and it is due to the vacuum fluctuations.

2. The transmon qubit Hamiltonian reads

$$\hat{H} = \omega_r \hat{a}^\dagger\hat{a} + \omega_t \hat{b}^\dagger\hat{b} - \frac{E_c}{2}\hat{b}^{\dagger 2}\hat{b}^2 - g(\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}). \tag{47}$$

The unperturbed Hamiltonian is given by the first three terms and its eigenstates read $|n,\mu\rangle$ where n labels the states of a linear oscillator and μ labels the states of a nonlinear oscillator. As before, the last term describe the interaction between the transmon and the cavity. We want to derive the effective Hamiltonian in the dispersive regime given by $|g/\Delta| \ll 1$ where $\Delta = \omega_t - \omega_r$, through a Schrieffer-Wolff transformation. Calling $\hat{V} = \hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}$ we have

$$\hat{S} = \sum_{m,\nu}\sum_{n,\mu} \frac{\langle m,\nu|\,\hat{V}\,|n,\mu\rangle}{E_{m,\nu} - E_{n,\mu}} |m,\nu\rangle\langle n,\mu| \,. \tag{48}$$

As before, we must compute the matrix elements of $\hat{V}$ in the unperturbed eigenbasis. We get

$$\begin{aligned}&\langle m,\nu|\,\hat{a}^\dagger\hat{b} + \hat{b}^\dagger\hat{a}\,|n,\mu\rangle = \langle m,\nu|\left(\sqrt{n+1}\sqrt{\nu}\,|n+1,\mu-1\rangle + \sqrt{n}\sqrt{\nu+1}\,|n-1,\mu+1\rangle\right)\\ &= \sqrt{n+1}\sqrt{\nu}\delta_{m,n+1}\delta_{\nu,\mu-1} + \sqrt{n}\sqrt{\mu+1}\delta_{m,n-1}\delta_{\nu,\mu+1},\end{aligned} \tag{49}$$

while the energy differences are given by

$$\begin{aligned}E_{n+1,\mu-1} - E_{n,\mu} &= -\Delta + E_c(\mu-1),\\ E_{n-1,\mu+1} - E_{n,\mu} &= \Delta - E_c\mu.\end{aligned} \tag{50}$$

From (49) and (50) we can write down the Schrieffer-Wolff generator as

$$\begin{aligned}\hat{S} &= \sum_{n,\mu}\left(\frac{\sqrt{\mu}\sqrt{n+1}}{-\Delta + E_c(\mu-1)}\,|n+1\rangle\langle n| \otimes |\mu-1\rangle\langle\mu| + \frac{\sqrt{n}\sqrt{\mu+1}}{\Delta - E_c\mu}\,|n-1\rangle\langle n| \otimes |\mu+1\rangle\langle\mu|\right)\\ &= \sum_{\mu}\left(\frac{\sqrt{\mu+1}\,|\mu+1\rangle\langle\mu|}{\Delta - E_c\mu}\,\hat{a} - \frac{\sqrt{\mu}\,|\mu-1\rangle\langle\mu|}{\Delta - E_c(\mu-1)}\,\hat{a}^\dagger\right).\end{aligned} \tag{51}$$

To derive the effective low-energy Hamiltonian we have to compute $[\hat{S},\hat{V}]$. Let's proceed by matrix multiplication

$$\begin{aligned}\hat{S}\hat{V} &= \sum_{\mu}\frac{\sqrt{\mu(\mu+1)}\,|\mu+1\rangle\langle\mu-1|}{\Delta - E_c\mu}\,\hat{a}^2 + \sum_{\mu}\frac{(\mu+1)\,|\mu+1\rangle\langle\mu+1|}{\Delta - E_c\mu}\,\hat{a}\hat{a}^\dagger\\ &= \sum_{\mu}\frac{\mu\,|\mu-1\rangle\langle\mu-1|}{\Delta - E_c(\mu-1)}\hat{a}^\dagger\hat{a} - \sum_{\mu}\frac{\sqrt{\mu(\mu+1)}\,|\mu-1\rangle\langle\mu+1|}{\Delta - E_c(\mu-1)}\,\hat{a}^{\dagger 2},\end{aligned} \tag{52}$$

while for the other term we obtain

$$\hat{V}\hat{S} = \sum_\mu \frac{\sqrt{(\mu+2)(\mu+1)}\,|\mu+2\rangle\langle\mu|}{\Delta - E_c\mu}\,\hat{a}^2 + \sum_\mu \frac{(\mu+1)\,|\mu\rangle\langle\mu|}{\Delta - E_c\mu}\,\hat{a}^\dagger\hat{a}$$
$$= \sum_\mu \frac{\mu\,|\mu\rangle\langle\mu|}{\Delta - E_c(\mu-1)}\hat{a}\hat{a}^\dagger - \sum_\mu \frac{\sqrt{\mu(\mu-1)}\,|\mu-2\rangle\langle\mu|}{\Delta - E_c(\mu-1)}\,\hat{a}^{\dagger 2}. \tag{53}$$

We immediately realize that the effective Hamiltonian contains counter-rotating terms proportional to $\hat{a}^2$ and $\hat{a}^{\dagger 2}$. Keeping only the diagonal terms and using the commutation relation $[\hat{a}, \hat{a}^\dagger] = \mathbb{1}$. This leads to the expression for the commutator

$$[\hat{S}, \hat{V}] = \sum_\mu \frac{2\mu\,|\mu\rangle\langle\mu|}{\Delta - E_c(\mu-1)} + \hat{a}^\dagger\hat{a}\sum_\mu \left(\frac{2\mu}{\Delta - E_c(\mu-1)} - \frac{2(\mu+1)}{\Delta - E_c\mu}\right)|\mu\rangle\langle\mu|\,. \tag{54}$$

The effective low-energy Hamiltonian is given by

$$\hat{H}_{\text{eff, cQED}} = \omega_r\hat{a}^\dagger\hat{a} + \sum_\mu\left(\omega_t + \frac{g^2}{\Delta - E_c(\mu-1)}\right)\mu\,|\mu\rangle\langle\mu| - \frac{E_c}{2}\hat{b}^{\dagger 2}\hat{b}^2$$
$$+ \hat{a}^\dagger\hat{a}\sum_\mu g^2\left(\frac{\mu}{\Delta - E_c(\mu-1)} - \frac{\mu+1}{\Delta - E_c\mu}\right)|\mu\rangle\langle\mu|\,, \tag{55}$$

which is the generalization of the dispersive Jaynes-Cumming Hamiltonian to general nonlinear multilevel systems.

3. If we truncate all the transmon energy levels but the first two we arrive at the effective Hamiltonian

$$\hat{H}_{\text{eff, cQED}} = \omega_r\hat{a}^\dagger\hat{a} + \left(\omega_t + \frac{g^2}{\Delta}\right)|1\rangle\langle 1| + g^2\left[-\frac{1}{\Delta}|0\rangle\langle 0| + \left(\frac{1}{\Delta} - \frac{2}{\Delta - E_c}\right)|1\rangle\langle 1|\right]\hat{a}^\dagger\hat{a}. \tag{56}$$

The third term of the above equation describe the dispersive light-matter interaction. If $E_c \to \infty$, it readily reduces to $(g^2/\Delta)\hat{\sigma}^z\hat{a}^\dagger\hat{a}$, which is the Jaynes-Cumming result. The fact that the transmon is a multilevel system with a finite Kerr nonlinearity E_c, *renormalizes* the dispersive interaction between the cavity and the qubit excited state by a factor $-2/(\Delta - E_c)$. To compute the effective dispersive shift when accounting for the multilevel structure of the transmon, we can write

$$\chi_{\text{cQED}} = \frac{1}{2}\left[\left(\frac{1}{\Delta} - \frac{2}{\Delta - E_c}\right) + \frac{1}{\Delta}\right] = \frac{g^2 E_c}{\Delta(\Delta - E_c)}. \tag{57}$$

For the Jaynes-Cumming model, we find instead $\chi_{\text{JC}} = g^2/\Delta$. Since $E_c \ll \Delta$, we have also $\chi_{\text{cQED}} \ll \chi_{\text{JC}}$.